Lecture 14 — 14.3 - Partial Derivatives

As usual, we will do things for functions of two variables first, then move on to more variables.

Def: The partial derivative of $f = f(x,y)$

- with respect to x at (a,b) is

$$\frac{\partial f}{\partial x}(a,b) = \frac{dg}{dx}(a) \quad , \text{where } g(x) = f(x,b).$$

Formally:

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

- with respect to y at (a,b) is formally

$$\frac{\partial f}{\partial y}(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

As in calc I, we can compute partial derivatives as functions of x & y (much as $f'(x)$ is a function of x). In this case,

- the partial derivative of f with respect to x is

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

: the partial derivative of f with respect to y is

$$\frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Note: There are a variety of notations for the partial derivative. If $z = f(x,y)$

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial x} = f_x = f_x(x,y) = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(f(x,y))$$

& similarly for $\frac{\partial f}{\partial y}(x,y)$.

So, how do we compute partial derivatives practically? Notice that, for example, in the definition of $\frac{\partial f}{\partial x}$, the y variable is completely ignored by h.

The result of this is that we are taking the derivative of f with respect to x while regarding y as simply a constant. The same is true of $\frac{\partial f}{\partial y}$; regard x as a constant and take the derivative with respect to y.

Ex: Find f_x and $f_y(1,4)$ where

$$f(x,y) = x^3 + 2x^2y + x^4y^2 + \sqrt{y}$$

Sol: To find f_x, treat y as a constant, and differentiate with respect to x:

$$f_x(x,y) = 3x^2 + 4xy + 4x^3y^2$$

($\sqrt{y}$ contributes nothing to f_x since, as far as x is concerned, y is a constant.)

Similarly, we find f_y:

$$f_y(x,y) = 2x^2 + 2x^4y + \frac{1}{2}\frac{1}{\sqrt{y}}$$

So, $$f_y(1,4) = 2(1)^2 + 2(1)^4(4) + \frac{1}{2}\frac{1}{\sqrt{4}} = 2 + 8 + \frac{1}{4}$$
$$= \frac{41}{4}.$$

◇

Back in calc I, we had an interpretation of the number $\frac{df}{dx}(a) = f'(a)$, namely, as the slope of the tangent line to $y=f(x)$ at $(a, f(a))$.

So, what does $f_x(a,b)$ and $f_y(a,b)$ represent?

Let's focus on $f_x(a,b)$. By observing the definition, we can say $f_x(a,b)$ is the rate of change of f in the x-direction at (a,b). This sounds a bit like "slope". How can we visualize $f_x(a,b)$? Here y is fixed to be b, so we look at the intersection of $z=f$ with $y=b$. This gives a curve, call it C_y, in the plane $y=b$. $f_x(a,b)$ is the slope of this curve at the point $(a,b,f(a,b))$ in this plane. See mathematica code for visuals.

Now, let's do some more derivatives.

Ex: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ where $f(x,y) = \sin(x\cos y)$.

Sol: First $\frac{\partial f}{\partial x}$: This requires the chain rule in one variable. Note that $\cos y$ is a constant as far as x is concerned, so

$$\frac{\partial f}{\partial x} = \cos(x\cos y)\frac{\partial}{\partial x}(x\cos y)$$

$$= \cos(x\cos y)\cdot(\cos y)$$

Now,

$$\frac{\partial f}{\partial y} = \cos(x\cos y)\cdot\frac{\partial}{\partial y}(x\cos y) = \cos(x\cos y)\cdot(-x\sin y)$$

$$= -\cos(x\cos y)(x\sin y).$$

$\diamond$

We can also do implicit differentiation. Assume z is implicitly defined as a function of x and y.

[redacted]

Ex: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where z is defined implicitly by $yz + x\ln y = z^2$.

Sol: Find $\frac{\partial z}{\partial x}$: take the derivative of both sides with respect to x:

$$y\frac{\partial z}{\partial x} + \ln y = 2z\frac{\partial z}{\partial x}$$

Now, solve for $\frac{\partial z}{\partial x}$: $(2z-y)\frac{\partial z}{\partial x} = \ln y$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{\ln y}{2z-y}$$

Similarly, we find $\frac{\partial z}{\partial y}$:

$$\overbrace{z + y\frac{\partial z}{\partial y}}^{\text{product rule}} + \frac{x}{y} = 2z\frac{\partial z}{\partial y}$$

Solve for $\frac{\partial z}{\partial y}$: $(2z-y)\frac{\partial z}{\partial y} = z + \frac{x}{y} = \frac{yz+x}{y}$

$$\Rightarrow \quad \frac{\partial z}{\partial y} = \frac{yz+x}{y(2z-y)}$$

◇

<u>More than two variables</u>:

The procedure is the same. To find, say, $\frac{\partial f}{\partial x}$ where $f = f(x,y,z)$, we treat y and z as constants and take the derivative with respect to x. The formal definition would be

$$\frac{\partial f}{\partial x}(x,y,z) = \lim_{h\to 0}\frac{f(x+h,y,z) - f(x,y,z)}{h}$$

Naturally, this goes on to as many variables as you want.

Ex: Find f_x, f_y, and f_z for $f(x,y,z) = ze^{xyz}$.

Sol: $\frac{\partial f}{\partial x} = yz^2e^{xyz}$, $\frac{\partial f}{\partial y} = xz^2e^{xyz}$

$$\frac{\partial f}{\partial z} = e^{xyz} + z(xye^{xyz}) = e^{xyz} + xyze^{xyz}$$

$\diamond$

Higher-Order derivatives

Naturally, we can take derivatives of derivatives, so why not partial derivatives of partial derivatives.

Notation:

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}, \text{ etc.}$$

notice how the order changes here

in f_{xy} it tells us to take the derivatives in a certain order. The notation $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ means "take the partial derivative of $\frac{\partial f}{\partial x}$ with respect to y"

We can keep going to third, fourth, fith, etc. partials..

<u>Ex</u>: Find all second partials of $f(x,y) = x^4y^3 - y^4$

<u>Sol</u>: $f_x = 4x^3y^3$, $f_y = 3x^4y^2 - 4y^3$

$f_{xx} = 12x^2y^3$, $f_{yy} = 6x^4y - 12y^2$

$\boxed{f_{xy} = 12x^3y \quad , \quad f_{yx} = 12x^3y}$

same!

$\diamond$

Notice that $f_{xy} = f_{yx}$ in this example. This is no coincidence! It is a manifestation of the following

<u>Clairaut's Theorem</u>: Suppose f is defined on a disk D containing the point. If f_{xy} and f_{yx} are both continuous on D then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Some definitions: Let f be a function.

$f = f(x,y)$	$f = f(x,y,z)$
The <u>gradient</u> is	The <u>gradient</u> is
$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$	$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

If $\vec{F}(x_1, x_2, \cdots, x_n)$ is a vector valued function of n variables, i.e.,

$$\vec{F}(x_1, x_2, \cdots, x_n) = \langle f_1(x_1, \cdots, x_n), \cdots, f_m(x_1, \cdots, x_n) \rangle$$

Then $$\frac{\partial \vec{F}}{\partial x_i} = \left\langle \frac{\partial f_1}{\partial x_i}, \frac{\partial f_2}{\partial x_i}, \cdots, \frac{\partial f_m}{\partial x_i} \right\rangle$$

Finally, the <u>total derivative of $\vec{F}$</u> is

$$D\vec{F}(x_1, \cdots, x_n) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

<u>Ex</u>: Consider the map $\vec{F}(r, \theta) = \langle r\cos\theta, r\sin\theta \rangle = \langle x, y \rangle$ which switches from polar to cartesian coordinates.

$$D\vec{F}(r, \theta) = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

One more object we will use next week.

For a function $f = f(x,y)$, the Hessian of f is

$$\mathrm{Hess}(f)(x,y) = \mathrm{Hess}(f) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$